

Two requested recurrence proofs

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Abstract

We prove the parity recurrence requested for the self-squaring triangle A078121 and the binomial recurrence requested for the position-colored composition sequence A382991. The latter follows from a formal-series identity that also gives every auxiliary array entry. We include a weighted extension and its first asymptotic correction. The conjectured recurrences, sequences, and their defining interpretations are credited to the original sources. Historical priority for these arguments has not been established.

1 A parity recurrence for A078121

Put $T(n, k) = 0$ for $k > n$. The triangle A078121 is determined by

$$T(n, 0) = 1, \quad T(n, k) = \sum_{i=0}^{n-1} T(n-1, i)T(i, k-1) \quad (1 \leq k \leq n). \quad (1)$$

Its diagonal entries are automatically one. The following formula was conjectured by user231922 in MathOverflow question 495545 [1] and is recorded in OEIS with credit to Mikhail Kurkov [2].

Theorem 1.1. *For $0 < k < n$,*

$$T(n, k) = \sum_{i=0}^{n-k-1} T(n-k, i)T(k+i, k-1)(1 + (-1)^{n+k+i+1}). \quad (2)$$

All matrices used below are infinite lower triangular matrices indexed by nonnegative integers. Every matrix entry in a product is a finite sum. Inverses of matrices with diagonal entries one are obtained recursively, and their entries likewise depend on finitely many entries. Thus no analytic convergence or operator-topology assertion is involved.

Let $M = (T(n, k))$ and define $S(A)(i, j) = A(i+1, j+1)$. Lower triangularity makes S multiplicative: the omitted summand in the product has a zero factor from the first row. Equation (1) is precisely

$$S(M) = M^2. \quad (3)$$

The first column and this identity determine M uniquely, because row n is computed from rows at most $n-1$.

Lemma 1.2. *If $D = \text{diag}(1, -1, 1, -1, \dots)$, then $M^{-1} = DMD$.*

Proof. Write $P_n(z) = \sum_{j=0}^n T(n, j)z^j$. The defining recurrence gives

$$P_0(z) = 1, \quad P_n(z) = 1 + z \sum_{i=0}^{n-1} T(n-1, i)P_i(z).$$

Induction shows that $P_n(-1) = 0$ for $n > 0$: only the $i = 0$ term survives, and $T(n-1, 0) = 1$. Hence $Ma = e_0$, where $a_i = (-1)^i$ and e_0 is the first coordinate vector. The first column of M^{-1} is therefore a .

Set $Q = DM^{-1}D$. Its first column is identically one. Since $S(D) = -D$, multiplicativity and (3) give

$$S(Q) = D S(M)^{-1} D = DM^{-2}D = Q^2.$$

The uniqueness just observed implies $Q = M$, proving the lemma. $\square$

Proof of Theorem 1.1. Fix $k \geq 1$ and put $r = 2^{k-1}$. Iterating (3) gives

$$S^{k-1}(M) = M^r, \quad S^k(M) = M^{2r}. \quad (4)$$

Separating the first row and column, and letting $\mathbf{1}$ denote the column of ones, gives the self-similar block decomposition

$$M = \begin{pmatrix} 1 & 0 \\ \mathbf{1} & M^2 \end{pmatrix}.$$

The first column of M^r below its top entry is consequently the vector

$$v = (I + M^2 + \dots + M^{2r-2})\mathbf{1} = (M + M^3 + \dots + M^{2r-1})e_0. \quad (5)$$

Here $Me_0 = \mathbf{1}$ was used in the second equality. Equation (4) identifies its entries as

$$v_i = (M^r)(i+1, 0) = T(k+i, k-1).$$

Multiplication of (5) by $M - M^{-1}$ telescopes:

$$(M - M^{-1})v = (M^{2r} - I)e_0. \quad (6)$$

Take coordinate $d = n - k > 0$. By (4), the right side is $T(n, k)$. Lemma 1.2 makes the left side

$$\sum_{i=0}^d T(d, i)T(k+i, k-1)(1 - (-1)^{d+i}).$$

The $i = d$ term is zero, and $1 - (-1)^{d+i} = 1 + (-1)^{n+k+i+1}$. This is (2). $\square$

For example, the initial rows are

n	$T(n, 0), \dots, T(n, n)$
0	1
1	1 1
2	1 2 1
3	1 4 4 1
4	1 10 16 8 1

At $(n, k) = (4, 2)$, only $i = 1$ contributes, giving $2T(2, 1)T(3, 1) = 2 \cdot 2 \cdot 4 = 16$. The boundary values $k = 0$ and $k = n$ must be retained separately.

2 A binomial recurrence for A382991

The next question [4] asks for a proof connecting

$$A(x) = 1 + \sum_{m \geq 1} \prod_{j=1}^m \left(jx + \frac{x^2}{1-x} \right) = \sum_{n \geq 0} a(n)x^n \quad (7)$$

to the array

$$R(0, k) = 1, \quad R(n, k) = \binom{n+k}{k} + \sum_{j=0}^{k+1} \binom{k+1}{j} R(n-1, j) \quad (n \geq 1). \quad (8)$$

The sequence and the generating function (7) predate this note and are recorded in OEIS A382991 [5].

Theorem 2.1. *For every $n \geq 1$, $a(n) = R(n-1, 0)$.*

All series in this section are formal power series. The factorial growth of $a(n)$ imposes no restriction on the coefficientwise operations below. For an indeterminate z , write $z^{\overline{m}} = z(z+1) \cdots (z+m-1)$, with $z^{\overline{0}} = 1$.

Lemma 2.2. *Let K be a commutative ring with identity and let $h \in 1 + xK[[x]]$. Put $B(n, k) = [x^n]h(x)^{k+1}$, and define*

$$S(0, k) = 1, \quad S(n, k) = B(n, k) + \sum_{j=0}^{k+1} \binom{k+1}{j} S(n-1, j) \quad (n \geq 1).$$

Then, for every $k \geq 0$,

$$\boxed{\sum_{n \geq 0} S(n, k)x^n = h \sum_{i \geq 0} x^i (h+1)^{\overline{i}} (h+i)^k.} \quad (9)$$

Proof. Let C_k be the right side of (9). Only $i \leq n$ can contribute to $[x^n]C_k$, so the expression is well defined. The binomial theorem yields

$$\begin{aligned} x \sum_{j=0}^{k+1} \binom{k+1}{j} C_j &= h \sum_{i \geq 0} x^{i+1} (h+1)^{\overline{i}} (h+i+1)^{k+1} \\ &= h \sum_{i \geq 0} x^{i+1} (h+1)^{\overline{i+1}} (h+i+1)^k \\ &= C_k - h^{k+1}. \end{aligned}$$

In the second line we used $(h+1)^{\overline{i}} (h+i+1) = (h+1)^{\overline{i+1}}$. Comparing coefficients gives the stated recurrence and initial row. They determine every successive row uniquely, proving the lemma. $\square$

Proof of Theorem 2.1. Take $h = u = (1-x)^{-1}$, so $[x^n]h^{k+1} = \binom{n+k}{k}$. Lemma 2.2, at $k = 0$, gives

$$\begin{aligned} 1 + x \sum_{n \geq 0} R(n, 0)x^n &= 1 + \sum_{m \geq 1} x^m u^{\overline{m}} \\ &= 1 + \sum_{m \geq 1} \prod_{j=1}^m x(u+j-1) \\ &= 1 + \sum_{m \geq 1} \prod_{j=1}^m \left(jx + \frac{x^2}{1-x} \right) = A(x). \end{aligned}$$

Coefficient comparison proves the claim. $\square$

The lemma also determines the full auxiliary array. Define nonnegative integers $d_{i,k,\ell}$ by

$$z(z+1)\cdots(z+i)(z+i)^k = \sum_{\ell=1}^{i+k+1} d_{i,k,\ell} z^\ell.$$

Since $[x^t](1-x)^{-\ell} = \binom{t+\ell-1}{\ell-1}$, equation (9) gives the finite expression

$$R(n, k) = \sum_{i=0}^n \sum_{\ell=1}^{i+k+1} d_{i,k,\ell} \binom{n-i+\ell-1}{\ell-1}. \quad (10)$$

For $i = k = 0$, the polynomial defining $d_{i,k,\ell}$ is z .

2.1 A weighted composition extension

For an integer $\beta \geq 0$, let $a_\beta(n)$ count compositions of n in which a singleton in position j has j possible colors, and every part of size at least two has β possible colors. Positions count parts from the left. The empty composition contributes one. Consequently

$$A_\beta(x) = \sum_{n \geq 0} a_\beta(n) x^n = 1 + \sum_{m \geq 1} \prod_{j=1}^m \left(jx + \frac{\beta x^2}{1-x} \right). \quad (11)$$

These definitions also make sense as polynomials in β , or as weighted counts for a real $\beta \geq 0$. At $\beta = 0$, any composition containing a longer part receives weight zero.

Put $B_\beta(0, k) = 1$ and, for $n \geq 1$, put

$$B_\beta(n, k) = \sum_{s=1}^{\min(n, k+1)} \binom{k+1}{s} \binom{n-1}{s-1} \beta^s.$$

Define $S_\beta(0, k) = 1$ and

$$S_\beta(n, k) = B_\beta(n, k) + \sum_{j=0}^{k+1} \binom{k+1}{j} S_\beta(n-1, j) \quad (n \geq 1).$$

Corollary 2.3. *For every $n \geq 1$, $a_\beta(n) = S_\beta(n-1, 0)$. More generally, with $h = 1 + \beta x/(1-x)$,*

$$\sum_{n \geq 0} S_\beta(n, k) x^n = h \sum_{i \geq 0} x^i (h+1)^{\bar{i}} (h+i)^k. \quad (12)$$

Proof. Expanding h^{k+1} shows that its coefficient of x^n is $B_\beta(n, k)$. Apply Lemma 2.2. At $k = 0$, multiplication by x and reindexing gives (11), since $x(h+j-1) = jx + \beta x^2/(1-x)$. $\square$

In particular, $\beta = 1$ recovers the requested array. When $\beta = 0$, equation (12) gives the explicit formula $S_0(n, k) = (n+1)!(n+1)^k$.

2.2 The first asymptotic correction

Write $H_m = \sum_{j=1}^m 1/j$. Compositions consisting only of singletons contribute $n!$. The first correction comes from one part of size two.

Theorem 2.4. *For each fixed real $\beta \geq 0$, as $n \rightarrow \infty$,*

$$\boxed{\frac{a_\beta(n)}{n!} = 1 + \frac{\beta H_{n-1}}{n} + O_\beta\left(\frac{(\log n)^2}{n^2}\right)}. \quad (13)$$

In particular, $a(n) \sim n!$.

Proof. A composition with $m = n - r$ parts has excess r above its length. If exactly s parts are longer than one, choose their positions and distribute r as positive excesses among those positions. Let e_s denote the elementary symmetric polynomial of degree s . This gives the exact identity

$$a_\beta(n) = n! + \sum_{r=1}^{n-1} (n-r)! \sum_{s=1}^{\min(r, n-r)} \binom{r-1}{s-1} \beta^s e_s\left(1, \frac{1}{2}, \dots, \frac{1}{n-r}\right). \quad (14)$$

Indeed, for a set $J \subseteq \{1, \dots, m\}$ of longer-part positions, the color weight is $\beta^{|J|} m! / \prod_{j \in J} j$. For $|J| = s$ there are $\binom{r-1}{s-1}$ positive excess distributions. This counts each composition once and proves (14).

The $r = 1$ term is exactly $\beta(n-1)! H_{n-1}$. For $2 \leq r \leq n/2$, use

$$\frac{(n-r)!}{n!} \leq \left(\frac{2}{n}\right)^r, \quad e_s\left(1, \frac{1}{2}, \dots, \frac{1}{n-r}\right) \leq H_n^s.$$

After division by $n!$, these remaining terms have sum at most

$$\begin{aligned} \sum_{r=2}^{\lfloor n/2 \rfloor} \left(\frac{2}{n}\right)^r \beta H_n (1 + \beta H_n)^{r-1} &\leq \frac{4\beta H_n (1 + \beta H_n)}{n^2} \frac{1}{1 - 2(1 + \beta H_n)/n} \\ &= O_\beta\left(\frac{(\log n)^2}{n^2}\right). \end{aligned}$$

The geometric-series bound holds for all sufficiently large n , and $H_n \leq 1 + \log n$.

For the tail $r > n/2$, put $B = \max(1, \beta)$. A composition with $m = n - r < n/2$ parts has weight at most $m! B^m$. There are $\binom{n-1}{m-1}$ uncolored compositions of length m , so its total normalized contribution is at most

$$\frac{\lfloor n/2 \rfloor!}{n!} B^{n/2} 2^{n-1} \leq B^{n/2} 2^{n-1} \left(\frac{2}{n}\right)^{\lceil n/2 \rceil}.$$

For fixed β , this decays faster than every power of $1/n$. Combining the $r = 1$ term with the two bounds proves (13). $\square$

3 Sources, checking, and attribution

The MathOverflow pages and direct Stack Exchange endpoints were checked on 19 October 2026. Both questions reported zero answers, and both answer endpoints returned empty lists. The power relation (4) is already recorded in OEIS and Epstein’s related notes [3]; the latter explicitly leave the June 2025 parity recurrence unresolved. Lemma 1.2 and (6) supply a proof here. No uniform speed comparison between implementations is asserted.

Exact A078121 checks covered 4,950 interior identities through $n = 100$, an independent reconstruction of 5,151 entries, direct inversion in dimension 24, and 1,331 matrix-power entries. Exact A382991 checks covered 5,985 auxiliary entries at $\beta \in \{0, 1, 2, 3, 5\}$, scalar identities through $n = 150$ at $\beta = 1$, literal enumeration through $n = 14$, and all 23 displayed OEIS terms. The scripts document the separate bounds for each check. No sequence extension is claimed, and numerical asymptotic diagnostics are not used as proof.

Li and Wang [6] use a different coloring convention, with colors according to part size at selected positions. Their inspected arXiv text does not mention A382991 or the requested recurrence. These searches do not establish historical priority.

Codex disclosure. OpenAI Codex generated the arguments, searches, code, and prose at Sulayman Bowles’s request. Separately prompted Codex agents reviewed the proofs and checked them with independently written exact programs. This is AI review, not independent human peer review or proof-assistant verification. The note does not represent a journal acceptance or posted MathOverflow answer. The original conjectures and sequence definitions are credited to their contributors.

References

- [1] user231922, “Equivalent recursion for A078121,” MathOverflow question 495545, 1 June 2025. <https://mathoverflow.net/questions/495545>.
- [2] P. D. Hanna and contributors, OEIS A078121, created 18 November 2002. Parity recurrence credited to M. Kurkov, 1 June 2025. <https://oeis.org/A078121>.
- [3] A. R. Epstein, *Sections of the binary partition function and the A078121 transfer matrix* and *The diagonals of A078121*, in *oeis-proofs*, revision feafd9b033ee5f4391b001d3dfd4d706b6ddf0ba, 16 August 2026. <https://github.com/Eppie/oeis-proofs>. The open-problem lists in `proofs/A000123.md`, section 8.2, and `proofs/A078121-diagonal.md`, section 6(c), include the June recurrence.
- [4] user231922, “Recursion for A382991,” MathOverflow question 491242, 18 April 2025. <https://mathoverflow.net/questions/491242>.
- [5] J. T. Rascoe and contributors, OEIS A382991, created 11 April 2025. <https://oeis.org/A382991>.
- [6] A. Li and H. Wang, *Combinatorics of positional colored compositions*, arXiv:2511.08529, 11 November 2025; *Integers* **26** (2026), A49. <https://arxiv.org/abs/2511.08529>.