

Lower bounds and position dependence in binary divisibility patterns

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Abstract

Let $N(w)$ be the least positive integer whose divisor indicator contains a binary word w , and $M(m) = \max_{|w|=m} N(w)$. We prove the superexponential lower bound

$$\log M(m) \geq m \log m - 2m \log \log m + (2 \log 2 - 1)m + o(m).$$

We also determine the exponential rate for every word with exactly one zero. Uniformly for nonnegative a, b with $a + b \rightarrow \infty$,

$$\log N(1^a 0 1^b) = b + \max(a, b) + o(a + b).$$

The second formula depends on the position of the zero and extends the endpoint-zero case. Matching upper bounds for the maximum and mean remain open. Historical priority is not claimed.

For $n \geq 1$, define $d_n(j) = 1$ if $j \mid n$, and $d_n(j) = 0$ otherwise. A binary word $w = w(0) \cdots w(m-1)$ occurs at $k \geq 1$ if $d_n(k+i) = w(i)$ for $0 \leq i < m$. Write $N(w)$ for the least such n . Ilkka Törmä asked for the growth of

$$M(m) = \max_{w \in \{0,1\}^m} N(w), \quad A(m) = 2^{-m} \sum_{w \in \{0,1\}^m} N(w)$$

over all words of length m [1]. Existence of $N(w)$ follows from the private-prime CRT construction in that question. We give a stronger lower bound for both quantities and a uniform result for the structured family

$$w_{a,b} = 1^a 0 1^b, \quad m = a + b + 1.$$

All logarithms are natural. We use the prime number theorem in the standard forms

$$\vartheta(x) = \sum_{p \leq x} \log p = x + o(x), \quad \psi(x) = \log \text{lcm}(1, \dots, \lfloor x \rfloor) = x + o(x),$$

and the consequence that the least prime strictly greater than t is $t + o(t)$. No assertion of novelty is made for these standard methods or the lcm bounds below.

1 A universal superexponential lower bound

Theorem 1. *If p is prime and $r \geq p$, then*

$$N(0^p 1^r) \geq r \binom{p^2 + r}{r}.$$

Consequently, for every sufficiently large integer m ,

$$\begin{aligned}\log M(m) &\geq m \log m - 2m \log \log m + (2 \log 2 - 1)m + o(m), \\ \log A(m) &\geq m \log m - 2m \log \log m + (\log 2 - 1)m + o(m).\end{aligned}$$

In particular, both logarithms have lower bound $(1 - o(1))m \log m$.

Proof. Suppose $0^p 1^r$ occurs at k , and put $u = k + p$. The one-positions force

$$D = \text{lcm}(u, u+1, \dots, u+r-1) \mid n.$$

This interval of r consecutive integers contains a multiple of every integer at most r , so $\text{lcm}(1, \dots, r) \mid D$. Among the preceding p zero entries there is an entry E divisible by p .

We claim $E \geq p^2$. Otherwise $E = pt$ with $1 \leq t < p$. Since p is prime, p and t are coprime; and both are at most r . Thus $pt \mid \text{lcm}(1, \dots, r) \mid n$, contradicting the zero at E . Since $E < u$, we have $u \geq p^2 + 1$.

The elementary lcm bound for r consecutive integers gives

$$n \geq D \geq \frac{u(u+1) \cdots (u+r-1)}{(r-1)!} = r \binom{u+r-1}{r} \geq r \binom{p^2+r}{r}.$$

The lcm inequality follows from Lemma 3 below, applied to the full interval of length r .

For the asymptotic statement, choose a prime

$$p \sim \frac{2m}{\log m}, \quad r = m - p.$$

Such a prime exists by the prime number theorem, and $r \geq p$ for all sufficiently large m . The weaker consequence of the finite bound

$$\log M(m) \geq 2r \log p - \log r! = r(2 \log p - \log r + 1) + o(m)$$

already suffices. Set $L = \log m$ and $c = pL/m = 2 + o(1)$. Then

$$\begin{aligned}r(2 \log p - \log r + 1) &= m \left(1 - \frac{c}{L}\right) (L - 2 \log L + 2 \log c - \log(1 - c/L) + 1) \\ &= mL - 2m \log L + (2 \log 2 - 1)m + o(m).\end{aligned}$$

This applies at every sufficiently large length m , not merely on a subsequence. Finally $A(m) \geq 2^{-m}M(m)$, which proves the bound for the arithmetic mean. $\square$

The question's displayed lower bound $M(m) \geq \frac{1}{2} \binom{2m}{m}$ is exponential in m . Theorem 1 gives the stronger superexponential lower bound above. It does not identify the correct order of either $\log M(m)$ or $\log A(m)$.

For a small exact example, take $p = r = 3$. The theorem gives $N(000111) \geq 3 \binom{12}{3} = 660$. Equality holds: the one-positions at 10, 11, 12 divide 660, while the preceding zero-positions 7, 8, 9 do not.

2 An exact feasibility criterion

Let $Q(x)$ be the largest prime power dividing x , with $Q(1) = 0$. For a proposed start k , put $x = k + a$ and

$$D_{a,b}(k) = \text{lcm}\{k + i : 0 \leq i < m, i \neq a\},$$

where an empty lcm is 1.

Lemma 2. *The word $w_{a,b}$ can occur at k if and only if*

$$Q(k + a) > \max(a, b).$$

When it can occur, the smallest witness at that start is $D_{a,b}(k)$. If $m \geq 2$, every feasible start satisfies

$$k \geq \max(1, b - a + 1).$$

Proof. The one-positions force $D_{a,b}(k) \mid n$. Thus an occurrence is possible exactly when $x \nmid D_{a,b}(k)$, and then $n = D_{a,b}(k)$ is the smallest witness. This failure of divisibility means that some prime power $q = p^{v_p(x)}$ dividing x divides no other block member.

The nearest other multiples of q are $x - q$ and $x + q$. These lie outside the block precisely when $q > a$ and $q > b$. If $q \leq a$, then $x - q$ is a positive block member; if $q \leq b$, then $x + q$ is a block member. This proves the criterion. Finally $x \geq Q(x) > \max(a, b)$, so $k = x - a \geq b - a + 1$, as well as $k \geq 1$. $\square$

Lemma 3. *For any nonempty subset S of an interval of m consecutive positive integers,*

$$\frac{\prod_{s \in S} s}{\text{lcm}(S)} \mid (m - 1)!.$$

In particular, for $m \geq 2$,

$$D_{a,b}(k) \geq \frac{k^{m-1}}{(m-1)!}.$$

Proof. Fix a prime p , and let c_j count members of S divisible by p^j . The p -adic valuation of the quotient is

$$\sum_{j \geq 1} (c_j - \mathbf{1}_{c_j > 0}).$$

An interval with diameter $m - 1$ contains at most $\lfloor (m - 1)/p^j \rfloor + 1$ multiples of p^j . Thus each summand is at most $\lfloor (m - 1)/p^j \rfloor$, and their sum is at most $v_p((m - 1)!)$. Applying the result to the $m - 1$ one-positions proves the last assertion. $\square$

The consecutive-integer lcm lower bound used in Törmä's question is credited there to Farhi [2]. Lemma 3 records the precise elementary variant needed for omitting a position and for a finite exact search.

3 The uniform asymptotic

Theorem 4. *Uniformly over $a, b \geq 0$ with $a + b + 1 = m \rightarrow \infty$,*

$$\log N(1^a 0 1^b) = b + \max(a, b) + o(m).$$

Equivalently, the maximum over these m pairs of the absolute error divided by m tends to zero. If $a/m \rightarrow \alpha \in [0, 1]$, then

$$\frac{\log N(1^a 0 1^b)}{m} \rightarrow \begin{cases} 2(1 - \alpha), & 0 \leq \alpha \leq \frac{1}{2}, \\ 1, & \frac{1}{2} \leq \alpha \leq 1. \end{cases}$$

Proof. Put $t = \max(a, b)$. Choose the least prime $p > t$, and set $k = p - a$. Since $p > a$, this is a positive start. The zero entry is p , and $p > t$, so Lemma 2 supplies the witness $D = D_{a,b}(k)$. Every selected entry is at most $p + b$, whence

$$\log N(w_{a,b}) \leq \log D \leq \psi(p + b) = t + b + o(m).$$

This error is uniform: $t \in [(m - 1)/2, m - 1]$, so the prime number theorem gives $p - t = o(m)$ throughout the range, and $p + b$ is between $m - O(1)$ and $2m + o(m)$.

For the lower bound, consider any feasible start k and put

$$F = \text{lcm}(k, k + 1, \dots, k + m - 1), \quad x = k + a.$$

Since $F = \text{lcm}(D, x) \leq Dx$, we have $D \geq F/x$. We handle all starts in three ranges.

If $k \leq m$, then $F = \text{lcm}(1, \dots, k + m - 1)$. Indeed, every integer less than k is at most $m - 1$ and has a multiple in the block of length m . Therefore, uniformly in this range,

$$\log D \geq \psi(k + m - 1) - \log(k + a) \geq k + m - 1 - o(m).$$

Lemma 2 gives $k \geq \max(1, b - a + 1)$. If $b \geq a$, this yields $k + m - 1 \geq 2b + 1$; if $a \geq b$, it yields $k + m - 1 \geq m$. Both exceed $b + t$, so $\log D \geq b + t - o(m)$.

If $m < k \leq 3m$, then F is divisible by $L_m = \text{lcm}(1, \dots, m)$ and by every prime in $(k - 1, k + m - 1]$. Those primes exceed m and hence are coprime to L_m . Consequently

$$\log F \geq \psi(m) + \vartheta(k + m - 1) - \vartheta(k - 1) = 2m + o(m).$$

All endpoint errors are uniformly $o(m)$, since the arguments lie between m and $4m$. Here $x \leq 4m$, so removing the zero entry costs only $O(\log m)$. As $b + t \leq 2m - 2$, the required lower bound follows.

Finally, if $k > 3m$, Lemma 3 and Stirling's formula give

$$\log D \geq (m - 1) \log k - \log((m - 1)!) \geq (1 + \log 3)m - O(\log m) > 2m - o(m).$$

This bound applies even to arbitrarily large k ; no boundedness assumption on $\log(k + a)$ is made in this range. Every witness at k is at least D . Thus all feasible starts satisfy the desired lower bound, uniformly in a, b . $\square$

Corollary 5. *For the family of all m single-zero words, the minimum has logarithm $(1 + o(1))m$, and the maximum has logarithm $(2 + o(1))m$. Their geometric mean G_m satisfies*

$$\log G_m = \left(\frac{5}{4} + o(1)\right)m.$$

Their arithmetic mean has logarithm $(2 + o(1))m$.

Proof. The first two statements follow from the minimum and maximum of the uniform rate profile. Its integral is

$$\int_0^{1/2} 2(1 - \alpha) d\alpha + \int_{1/2}^1 1 d\alpha = \frac{5}{4},$$

which gives the geometric mean by a Riemann sum and uniform error control. The arithmetic mean lies between the maximum divided by m and the maximum. $\square$

These means are over the restricted single-zero family, not over all 2^m binary words in the original question.

4 Exact computations and scope

The accompanying verifier computes the exact minima for all 5,050 single-zero words of lengths 1 through 100. A prime at the zero entry gives a finite initial witness. At every possible start k , prefix and suffix lcms give all m omitted-position lcms. Lemma 3 proves the exact stopping rule: if

$$k^{m-1} > U(m-1)!,$$

then no start at or beyond k can improve a current witness of size U . All comparisons establishing minimality use integers. The program separately verifies the prime-power criterion at every scanned start and checks 101,800 instances of the subset-lcm divisibility. A separately written prime-valuation implementation matches all 464 nontrivial family values through length 30 and checks literal divisor indicators through length 6. The proofs do not depend on these finite checks.

Theorem 4 determines a larger structured family than the endpoint word 01^{m-1} . It does not determine the growth of the maximum or arithmetic mean over all words. In particular, words with two zeros present additional interactions between the excluded positions; the single-zero criterion must not be applied to them separately without accounting for the other excluded entry.

Authorship, tools, and review. Prepared for Sulayman Bowles using OpenAI Codex. Codex developed the arguments, wrote this exposition and the programs, and performed separately prompted proof reviews and exact computations. Such review is not human peer review or formal verification. A focused search of the original question and related lcm terminology did not locate the stated position-dependent formula; historical priority remains unconfirmed. The prime number theorem and elementary lcm methods are standard inputs. No external posting was performed by this research subtask.

References

- [1] I. Törmä, *Occurrence of binary words in divisibility patterns*, MathOverflow question 372973, 30 September 2020.
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- [2] B. Farhi, *Minorations non triviales du plus petit commun multiple de certaines suites finies d'entiers*, C. R. Math. Acad. Sci. Paris **341** (2005), no. 8, 469–474. doi:10.1016/j.crma.2005.09.019.