

Solutions to two requested mathematical problems

Sulayman Bowles

19 October 2026

Research initiated and directed by the named author; mathematical development, computations and drafting assisted by OpenAI Codex.

1. Every ordering from binary-coefficient polynomials

The request. On 8 April 2025, MathOverflow user ABB asked whether, for every positive integer N , there is one fixed tuple of distinct real numbers on which polynomials with coefficients in $\{0, 1\}$ and zero constant term realize every permutation as the order of their values [1]. The retrieved question and the Stack Exchange API displayed no answers when checked on 19 October 2026. The theorem below gives an affirmative answer. All inequalities are strict, so no convention for ties is needed.

Theorem 1. *For every $N \geq 1$, there are distinct $s_1, \dots, s_N \in (-1, 0)$ such that for every permutation $r = (r_1, \dots, r_N)$ of $(1, \dots, N)$, some polynomial $p_r(x) = \sum_{k \geq 1} a_k x^k$, with $a_k \in \{0, 1\}$ and only finitely many nonzero coefficients, satisfies*

$$r_i < r_j \implies p_r(s_i) < p_r(s_j).$$

The nodes may also be chosen rational.

The idea is to interpolate the desired ranks by an integer polynomial. Near negative inputs of magnitude one, even and odd powers provide positive and negative contributions. A block of distinct monomials can therefore encode an arbitrary integer coefficient while keeping every coefficient of the resulting polynomial equal to zero or one.

Lemma 2. *Let $q(t) = \sum_{j=0}^d c_j t^j \in \mathbb{Z}[t]$, with $C = \max_j |c_j| \geq 1$ and $B = \sum_j |c_j|$. For an even integer $M > 2C$, set*

$$p_{q,M}(x) = \sum_{c_j > 0} \sum_{\ell=1}^{c_j} x^{jM+2\ell} + \sum_{c_j < 0} \sum_{\ell=1}^{-c_j} x^{jM+2\ell-1}. \quad (1)$$

Then $p_{q,M}$ has coefficients in $\{0, 1\}$ and zero constant term. If $1/(N+1) \leq t < 1$ and $s = -t^{1/M}$, then

$$|p_{q,M}(s) - q(t)| \leq \frac{2CNB}{M}.$$

Proof. All exponents in (1) are positive. For each j they lie in $(jM, jM + 2C]$, and these intervals do not overlap. Each exponent occurs once, proving the coefficient claim.

Since M is even, $s^{jM} = t^j$. An even offset contributes $+t^j t^{2\ell/M}$ and an odd offset contributes $-t^j t^{(2\ell-1)/M}$, exactly the sign required by c_j . For $\alpha > 0$ and $0 < t \leq 1$, convexity of $u \mapsto u^{-\alpha}$ at $u = 1$ gives

$$0 \leq 1 - t^\alpha \leq \alpha(t^{-1} - 1).$$

Here $t^{-1} - 1 \leq N$, and each offset is at most $2C$. Thus each of the B encoded monomials differs from its intended signed contribution $\pm t^j$ by at most $2CN/M$. Summation proves the estimate. $\square$

2. One common set of nodes for every permutation

Proof of Theorem 1. Put $t_i = i/(N+1)$ and $D = (N-1)!$. Given a rank vector r , define

$$q_r(t) = \sum_{i=1}^N r_i (-1)^{N-i} \binom{N-1}{i-1} \prod_{\substack{1 \leq h \leq N \\ h \neq i}} ((N+1)t - h).$$

This polynomial has integer coefficients. At $t = t_i$, all summands except the i th vanish, and the remaining product is $(-1)^{N-i}(i-1)!(N-i)!$. Consequently

$$q_r(t_i) = Dr_i.$$

There are finitely many rank vectors. Let C be the largest absolute coefficient of their interpolation polynomials, and B the largest sum of absolute coefficients. Both are positive integers. Choose

$$M = 8NCB, \quad s_i = -t_i^{1/M}.$$

This common M is even and exceeds $2C$. Applying Lemma 2 to every q_r yields

$$|p_{q_r, M}(s_i) - Dr_i| \leq \frac{1}{4} \quad (1 \leq i \leq N).$$

Thus whenever $r_i < r_j$,

$$p_{q_r, M}(s_j) - p_{q_r, M}(s_i) \geq D - \frac{1}{2} > 0.$$

The same nodes work for all permutations.

There are only finitely many strict inequalities in this construction. By continuity they remain true when each node is perturbed sufficiently little. Disjoint rational approximations to the nodes therefore give distinct rational nodes with the same property. $\square$

Nodes explicit in N . The definition above uses maxima over a finite family, but this enumeration can be avoided. Put

$$H = N(4N+2)^{N-1}, \quad M = 8NH^2, \quad s_i = -\left(\frac{i}{N+1}\right)^{1/M}.$$

These nodes satisfy the theorem. Indeed, the sum of the absolute coefficients in the i th interpolation basis polynomial is

$$\binom{N-1}{i-1} \prod_{h \neq i} (N+1+h) \leq \binom{N-1}{i-1} (2N+1)^{N-1}.$$

Multiplying by $r_i \leq N$ and summing gives $B \leq H$, and hence $C \leq H$. The same error estimate is at most $2NH^2/M = 1/4$. The constructed polynomials have degree at most $(N-1)M + 2H$.

For $N = 1$, the construction is still valid: $q(t) = 1$, $M = 8$ and $p(x) = x^2$. The bounds are sufficient existence bounds; no degree optimality is claimed.

3. A small exact example and verification

For three inputs, much smaller witnesses suffice. Fix

$$s_1 = -\frac{9}{10}, \quad s_2 = -\frac{4}{5}, \quad s_3 = -\frac{7}{10}.$$

In the table, an ordering (i, j, k) means $p(s_i) < p(s_j) < p(s_k)$.

Ordering	Polynomial
(1, 2, 3)	x
(1, 3, 2)	$x^4 + x^5$
(2, 1, 3)	$x^5 + x^6$
(2, 3, 1)	$x^3 + x^4 + x^5 + x^6$
(3, 1, 2)	$x^4 + x^5 + x^6 + x^7$
(3, 2, 1)	$x + x^2$

For example, $p = x^4 + x^5$ gives

$$p(s_1) = \frac{6561}{100000}, \quad p(s_2) = \frac{8192}{100000}, \quad p(s_3) = \frac{7203}{100000},$$

and therefore realizes $(1, 3, 2)$. All six rows were checked using exact rational arithmetic. The smallest adjacent gap across the table is $321/50000 > 0$.

An additional exact certificate uses the four rational nodes

$$\left(-\frac{19}{20}, -\frac{17}{20}, -\frac{3}{4}, -\frac{13}{20}\right).$$

All 24 orderings are realized by polynomials of degree at most 16; the archive records every polynomial and its rational values. No minimality claim is made for either example.

Executed checks. The constructive verifier examined all 873 rank vectors for $N = 1, \dots, 6$, checking exact integer interpolation, signs, exponent separation and the uniform error bound. High-precision evaluations were additional diagnostics rather than the basis of the proof. A separately written verifier checked every target permutation for $N \leq 4$ by exact rational enclosing intervals, without numerical evaluation of the algebraic nodes. The compact three- and four-node examples were independently rechecked with rational arithmetic.

Status. This is a complete affirmative answer to the stated question. It uses elementary interpolation and coefficient encoding. The comparison with the inspected request does not establish historical priority over all literature. No public answer or external submission has been made.

4. A second requested proof: a continued-fraction recursion

On 10 April 2025, MathOverflow user231922 asked for a proof of the following identity [4]. Define $R(0, k) = 1$ for $k \geq 0$ and

$$R(n, k) = R(n-1, k+1) + \sum_{i=0}^{n-1} \sum_{j=0}^k \binom{k}{j} R(n-i-1, j) R(i, k-j).$$

The conjecture is $R(n, 0) = a(n)$, where

$$\sum_{n \geq 0} a(n)x^n = \frac{1}{1-x - \frac{x}{1-2x - \frac{x}{1-3x - \dots}}}.$$

The sequence is OEIS A302285 [3]. Its continued fraction and related combinatorics predate this request; the result here addresses the displayed double-binomial recurrence.

Proof. Introduce a second variable t and, for $h \geq 0$, define formal continued-fraction tails by

$$F_h = \frac{t}{1 - (h+1)x - xF_{h+1}}, \quad F_h = t + O(x).$$

They are well defined in $t\mathbb{Z}[t][[x]]$. To determine the coefficient of x^n , only finitely many further tails are needed; changing a tail introduces a further factor of x at each preceding level.

Set

$$E_h = xt \frac{\partial F_h}{\partial t} - (1 - hx)F_h + t + xF_h^2.$$

Differentiating the tail equation and substituting it again gives

$$E_h = xF_h(F_h - F_{h+1}) + x^2 F_h^2 \frac{\partial F_{h+1}}{\partial t} = \frac{x F_h^2}{t} E_{h+1}.$$

Here F_h^2/t is a formal series with polynomial coefficients, so no inversion of t is needed. Iterating the identity shows that E_h is divisible by x^m for every m . Hence $E_h = 0$.

Write $F_0 = \sum_{n \geq 0} P_n(t)x^n$. Taking coefficients gives

$$P_0 = t, \quad P_n = tP'_{n-1} + \sum_{i=0}^{n-1} P_{n-i-1}P_i \quad (n \geq 1).$$

Let $D = t d/dt$ and $S(n, k) = (D^k P_n)(1)$. The binomial Leibniz rule for the derivation D gives exactly the displayed recurrence for S . Also $S(0, k) = 1$. Induction on n therefore gives $S(n, k) = R(n, k)$. Finally $F_0(x, 1)$ is the required continued fraction, and so

$$\boxed{R(n, 0) = P_n(1) = a(n) \quad (n \geq 0).}$$

□

5. Consequences, checks, and attribution

The polynomial recurrence gives more than the requested first column. Writing $P_n(t) = \sum_{m=1}^{n+1} c_{n,m} t^m$, one obtains

$$R(n, k) = \sum_{m=1}^{n+1} c_{n,m} m^k.$$

Thus an infinite row in k is represented by a finite polynomial. For example,

$$P_0 = t, \quad P_1 = t + t^2, \quad P_2 = t + 4t^2 + 2t^3,$$

$$P_3 = t + 11t^2 + 16t^3 + 5t^4.$$

The original implementation checked all 1,081 recurrence pairs with $n + k \leq 45$, and compared independent continued-fraction inversion with direct colored Schröder-path enumeration through $n = 150$. A separate reviewer checked all 2,556 pairs with $n + k \leq 70$, the full polynomial continued fraction through $n = 22$, and all 23 terms displayed in the inspected OEIS entry. Every comparison agreed exactly.

Alexander Burstein’s comment on the original question supplies the colored Schröder-path interpretation. The surrounding T-fraction theory appears in work of Bitonti, Deb and Sokal [2]; their Table 1 on page 58 includes A302285. These sources are prior context, and no new general theory of continued fractions is claimed here.

Assistance and limitations. OpenAI Codex was used to find the requests, develop the arguments, write and execute the verification programs, search sources, and draft the notes. Separately prompted Codex instances reviewed the proofs and wrote independent exact checks. These are automated reviews, not formal proof-assistant verification or external peer review. The named author initiated and directed this work. Neither solution has been posted externally. Historical priority is not established.

References

- [1] ABB, *On the coverage of permutations by argsort of polynomial evaluations*, MathOverflow, 8 April 2025.
<https://mathoverflow.net/questions/490705/>.
- [2] V. Bitonti, B. Deb and A. D. Sokal, *Thron-type continued fractions (T-fractions) for some classes of increasing trees*, arXiv:2412.10214v1 (2024), Table 1, p. 58.
<https://arxiv.org/abs/2412.10214>.
- [3] The OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, A302285. <https://oeis.org/A302285>.
- [4] user231922, *Recursion for A302285*, MathOverflow, 10 April 2025.
<https://mathoverflow.net/questions/490843/>.

All linked question and sequence pages were inspected on 19 October 2026.